Quantum Entanglement of Photons in Doubled q-Fock Space

He-Shan Song*, Dong Mi†, Ling Zhou and Chong Li

Department of Physics, Dalian University of Technology,

Dalian 116024, P. R. China

A doubled q-Fock space is constructed by introducing an idle mode system dual to the physical one under consideration. The quantum entangtlements of photons in the squeezed states and thermal states based on the douled q-Fock space are discussed.

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1. Introduction

The concept of the quantum entanglement which can not be factorized into a product of one-particle state was proposed by Schrödinger in 1935 [1]. The quantum entanglement may be considered as most important and intriguing character of quantum mechanics. In recent years, special attentions have been paid to study quantum entangled states. Some quantum optical experiments have already prepared quantum entangled states. A very convenient method for preparing entangled photon pair by parametric down-conversion in laser pumped non-linear crystal was discovered by Burnham and Weinberg in 1970. Their discovery permitted the development of two-photons interferometry by many authors[2]. By applying finite resolution measurement, the non-classical correlations of entangled photons are analyzed in the reference [3]. It has been demonstrated that the quantum entanglement is key process in the quantum computation, quantum teleportation, quantum cryptography and so on.

On the other hand, the study of exactly solvable statistical models has led to a new algebra, which is deformation of universal enveloping algebra. These deformed algebraic structures are now usually called quantum group [4-6]. In recent years, considerable interest in mathematical physics has concentrated on the quantum group and corresponding algebras

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*E-mail: hssong@dlut.edu.cn

†E-mail: dongmi@dlut.edu.cn

[7-9]. Of particular interest is the Bose realization of the quantum harmonic oscillators, which play an important roles in the study of quantum group and its applications in physics. For a system of q-deformed bosonic oscillators (for a system of photons, for example), since the usual commutation relations (Heisenberg-Weyl algebra) are replaced by the q-deformed one, the physical properties of system will be affected by the deformation parameter q.

The purpose of present paper is to investigate some quantum properties such as squeezing, quantum entanglement including thermal entanglement of photons in the doubled q-Fock space. In the section 2, we will construct a doubled q-Fock space by introducing an idle mode system dual to the real one. In section 3, we will discuss the entanglement of two-mode photons with infinite degree of freedom in the doubled q-Fock space. In the section 4, we will define a q-deformed thermal vacuum state and show the thermal vacuum is in fact a thermally entangled state. Some conclusions are summarized at the end of the paper.

2. The doubled q-Fock space

As is well known, an assembly of photons is simplest example of a Bose system of independent particles. Photons are quanta of the electromagnetic field. An electromagnetic field with frequency ω is equivalent to a bosonic oscillators with frequency ω and the n-th excited level of bosonic oscillator corresponds to the state of the electromagnetic field with n-photons, each of which has the energy $\epsilon = \hbar\omega$. So a state of an electromagnetic field can be denoted by the number state $|n\rangle$ with $n = 0, 1, 2, \cdots$. All states $\{|n\rangle, n = 0, 1, 2, \cdots \}$ construct a Fock space. Different states in Fock space are connected by creation and annihilation operators of photons.

Now we introduce operators $\{a, a^+, N, 1\}$ of q-deformed bosonic oscillator and define, analogue to the usual bosonic oscillators, q-deformed Fock space [10]

$$\left\{ |n\rangle = (D_q(n))^{-\frac{1}{2}} (a^+)^n |0\rangle, n = 0, 1, 2, \cdots \right\} \tag{2.1}$$

based on the vacuum state defined as $a|0\rangle = 0$, where a denotes q-deformed annihilation operator of photon. The deformation function $D_q(n)$ in the equation (2.1) satisfies

$$D_q(n)|_{q\to 1} = n, \tag{2.2}$$

$$D_q(0) = 0, \quad D_q(1) = 1 \tag{2.3}$$

and

$$D_q(n)! = \prod_{k=1}^{n} D_q(k). \tag{2.4}$$

Analogue to the usual case, the action of annihilation (creation) operator a (a^+) and number operator N on the state $|n\rangle$ can be defined as

$$a\,|n\rangle = \sqrt{D_q(n)}\,|n-1\rangle, \tag{2.5}$$

$$a^+\,|n\rangle = \sqrt{D_q(n+1)}\,|n+1\rangle, \tag{2.6}$$

$$N\,|n\rangle = n\,|n\rangle. \tag{2.7}$$

Now we introduce an operator P_{mn} defined as

$$P_{mn} = |m\rangle\langle n| \tag{2.8}$$

with $m, n = 0, 1, 2, \cdots$. The operator P_{mn} is an infinite dimensional matrix in which the element at m-th row n-th column is one and the others are zero. By using the operator P_{mn}, one can immediately find the matrix representations of the operators $\{a, a^+, N, 1\}$:

$$a = \sum_{n=0}^{\infty} \sqrt{D_q(n+1)}P_{n,n+1}, \tag{2.9}$$

$$a^+ = \sum_{n=0}^{\infty} \sqrt{D_q(n+1)}P_{n+1,n}, \tag{2.10}$$

$$N = \sum_{n=0}^{\infty} nP_{nn}, \tag{2.11}$$

$$1 = \sum_{n=0}^{\infty} P_{nn}. \tag{2.12}$$

From equation (2.9) to equation (2.12) we find

$$a^+a = \sum_{n=0}^{\infty} D_q(n)P_{nn} = D_q(N), \tag{2.13}$$

$$aa^+ = \sum_{n=0}^{\infty} D_q(n+1)P_{nn} = D_q(N+1) \tag{2.14}$$

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and so we obtain the generalized q-Heisenberg-Weyl algebra

$$\left[a, a^+\right] = D_q(N+1) - D_q(N), \tag{2.15}$$

$$\left[N, a^+\right] = a^+, \tag{2.16}$$

$$[N, a] = -a. \tag{2.17}$$

Obviously,

$$D_q(N)\left|n\right\rangle = \sum_{m=0}^{\infty} D_q(m)P_{mm}\left|n\right\rangle = D_q(n)\left|n\right\rangle, \tag{2.18}$$

$$D_q(N+1)\left|n\right\rangle = \sum_{m=0}^{\infty} D_q(m+1)P_{mm}\left|n\right\rangle = D_q(n+1)\left|n\right\rangle \tag{2.19}$$

and so the function $D_q(n)$ is eigenvalue of the operator $D_q(N)$.

Note that the deformation function $D_q(x)$ is not unique in general. The form of $D_q(x)$ will be related with Bose realization scheme of $SU(2)$. Different deformation functions will give different deformation algebras. It is easy to show that deformation function $D_q(n) = \frac{q^n - q^{-n}}{q - q^{-1}}$, for example, leads to the Biedenharn [6] and Mackfalan's [5] deformation algebra

$$aa^+ - qa^+a = q^{-N}. \tag{2.20}$$

Based on the q-Fock space, one can also construct doubled q-Fock space. Since in quantum optics we often need consider two mode system such as non-degenerate parametric amplifier gives rising to two mode squeezing and the doubled Fock space has to be considered. In order to construct doubled q-Fock space, it is convenient to introduce an idle mode system which is of exactly same structure as the physical one under consideration. We will denote the quantities associated with the idle mode photons by tilde such as idle photon number state by $|\tilde{n}\rangle$, annihilation (creation) operator of an idle photon by $\tilde{a}$ ($\tilde{a}^+$). Then the doubled q-Fock space spanned by direct product $|n\rangle \otimes |\tilde{n}\rangle \equiv |n\tilde{n}\rangle$ can be defined as

$$\left\{ |n\tilde{n}\rangle = \frac{(a^+)^n(\tilde{a}^+)^n}{D_q(n)!}\left|0\tilde{0}\right\rangle, n = 0, 1, 2, \cdots \right\}, \tag{2.21}$$

where the tilde operators $\tilde{a}$, $\tilde{a}^+$ satisfy the same algebra as corresponding non-tilde operators due to the duality of q-Fock space [11]. That is

$$\left[\tilde{a}, \tilde{a}^+\right] = D_q(\widetilde{N}+1) - D_q(\widetilde{N}), \tag{2.22}$$

$$\left[\widetilde{N}, \tilde{a}^+\right] = \tilde{a}^+, \tag{2.23}$$

$$\left[\widetilde{N}, \tilde{a}\right] = -\tilde{a}. \tag{2.24}$$

In the doubled q-Fock space, the expectation value of a bose-like operator F is given by

$$\langle \widetilde{m}n| F |n'\widetilde{m}'\rangle = \langle n| F |n'\rangle \, \delta_{mm'} \tag{2.25}$$

and that of corresponding $\widetilde{F}$ is

$$\langle \widetilde{m}n| \widetilde{F} |n'\widetilde{m}'\rangle = \langle \widetilde{m}| F |\widetilde{m}'\rangle \, \delta_{nn'}. \tag{2.26}$$

The actions of operators $\tilde{a}$, $\tilde{a}^+$ and $\widetilde{N}$ on the doubled q-Fock state $|n\tilde{n}\rangle$ are defined, analogue to the equation (2.5) to equation (2.7), as

$$\tilde{a} |n\tilde{n}\rangle = \sqrt{D_q(n)} |n, \tilde{n}-1\rangle, \tag{2.27}$$

$$\tilde{a}^+ |n\tilde{n}\rangle = \sqrt{D_q(n+1)} |n, \tilde{n}+1\rangle, \tag{2.28}$$

$$\widetilde{N} |n\tilde{n}\rangle = n |n\tilde{n}\rangle. \tag{2.29}$$

Having defined the doubled q-Fock space, we can discuss some quantum properties of photons in the doubled q-Fock space. In the subsequent sections, we will investigate some quantum properties such as squeezing, entanglement including thermal entanglement of photons in the doubled q-Fock space.

3. Entanglement of photons in squeezed states

Based on the above discussions, one can define squeezed vacuum state in the doubled q-Fock space as

$$|O(\xi)\rangle = \exp\left\{\xi\left(a^+\tilde{a}^+ - a\tilde{a}\right)\right\} |0\tilde{0}\rangle = \sum_{n=0}^{\infty} P_n^{\frac{1}{2}} |n\tilde{n}\rangle, \tag{3.1}$$

where $|n\tilde{n}\rangle$ is given by the equation(2.21) and

$$P_n = \frac{\tanh^{2n} \xi}{\cosh^2 \xi} \tag{3.2}$$

denotes probability of $2n$-photons state. The squeezing parameter ξ is taken to be real for simplicity. It is easy to verify

$$\sum_{n=0}^{\infty} P_n = \frac{1}{\cosh^2 \xi} \sum_{n=0}^{\infty} \tanh^{2n} \xi = 1 \tag{3.3}$$

and so the squeezed vacuum is normalized as

$$\langle O\left(\xi\right)| \left|O\left(\xi\right)\right\rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_m^{\frac{1}{2}} P_n^{\frac{1}{2}} \langle n\widetilde{n}| \left|m\widetilde{m}\right\rangle = \sum_{n=0}^{\infty} P_n = 1. \tag{3.4}$$

From the equation (3.1) we see that the squeezed vacuum state $\left|O\left(\xi\right)\right\rangle$ is an infinite superposition of doubled number states $\left|n\widetilde{n}\right\rangle \equiv \left|n\right\rangle \otimes \left|\widetilde{n}\right\rangle$. The state $\left|O\left(\xi\right)\right\rangle$ can not be factorized into a direct product of one-mode states. As photons are created in pair, there is perfect correlation between the photons in each state. So we can regard the q-squeezed vacuum $\left|O\left(\xi\right)\right\rangle$ as an example of partly entangled state for two-mode with $2n$-photons. Its entanglement is given by Shannon's quantum entropy

$$\begin{aligned}
E &= -\sum_{n=0}^{\infty} P_n \log_2 P_n \\
&= -\sum_{n=0}^{\infty} \frac{\tanh^{2n} \xi}{\cosh^2 \xi} \ln \frac{\tanh^{2n} \xi}{\cosh^2 \xi} \\
&= -\left\{ \sinh^2 \xi \log_2 \sinh^2 \xi - (1 + \sinh^2 \xi) \log_2(1 + \sinh^2 \xi) \right\} \\
&= -\left\{ \overline{n_0} \log_2 \overline{n_0} - (1 + \overline{n_0}) \log_2(1 + \overline{n_0}) \right\},
\end{aligned} \tag{3.5}$$

where $\overline{n_0} = \sinh^2 \xi$ denotes the average number of photons in the undeformed squeezed state, and k is Boltzmann constant.

The average number of photons in the q-squeezed vacuum is given by

$$\overline{n} = \langle O\left(\xi\right)| a^+ a \left|O\left(\xi\right)\right\rangle = \sum_{n=0}^{\infty} D_q(n) P_n. \tag{3.6}$$

If we take the Biedenharn and Mackfalan's deformation scheme

$$D_q(n) = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{3.7}$$

and substituting $D_q(n)$ into the equation (3.6), we obtain

$$\overline{n} = \frac{\tanh^2 \xi}{\cosh^2 \xi} \left\{ \frac{C_1}{1 - q \tanh^2 \xi} + \frac{C_2}{1 - q^{-1} \tanh^2 \xi} \right\} \tag{3.8}$$

with

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$$C_1 = \frac{q}{q - q^{-1}}, \quad C_2 = \frac{q^{-1}}{q^{-1} - q}, \tag{3.9}$$

$$C_1 + C_2 = 1 . \tag{3.10}$$

It is easy to see that when $q \to 1$, the equation (3.8) reduces to $\overline{n} = \sinh^2 \xi$, which is just the average number of photons in undeformed squeezed state.

We see from above discussions that the entanglement of two-photon is dependent only on the parameter ξ but independent of the deformation parameter q. The average photon number, however, depended on both of parameters ξ and q as seen from equation (3.8). The average number of photons deviated from $\sinh^2 \xi$ and divided into two parts due to q-deformation of the Fock space.

Consider now the quantum fluctuations of two quadratures U_1 and U_2. The quadratures are defined as

$$U_1 = (a + a^+ + \tilde{a} + \tilde{a}^+)/2^{\frac{3}{2}} , \tag{3.11}$$

$$U_2 = (a - a^+ + \tilde{a} - \tilde{a}^+)/2^{\frac{3}{2}}i. \tag{3.12}$$

Then the quantum fluctuations of U_i $(i = 1, 2)$ in the q-squeezed vacuum are given by

$$(\Delta U_i)^2 = \langle O\left(\xi\right)| U_i^2 |O\left(\xi\right)\rangle - (\langle O\left(\xi\right)| U_i |O\left(\xi\right)\rangle)^2 . \tag{3.13}$$

It is easy to calculate the vacuum expectation values

$$\langle O\left(\xi\right)| a^+ a |O\left(\xi\right)\rangle = \overline{n}, \tag{3.14}$$

$$\langle O\left(\xi\right)| a a^+ |O\left(\xi\right)\rangle = \overline{n}/\tanh^2 \xi, \tag{3.15}$$

$$\langle O\left(\xi\right)| a \tilde{a} |O\left(\xi\right)\rangle = \overline{n}/\tanh \xi, \tag{3.16}$$

$$\langle O\left(\xi\right)| \tilde{a} \tilde{a}^+ |O\left(\xi\right)\rangle = \overline{n}/\tanh \xi \tag{3.17}$$

and the others are zero. Substituting these results to the equation (3.13) we get

$$(\Delta U_1)^2 = \frac{1}{4} \frac{(1 + \tanh \xi)^2}{\tanh^2 \xi} \overline{n} , \tag{3.18}$$

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$$(\Delta U_2)^2 = \frac{1}{4} \frac{(1 - \tanh \xi)^2}{\tanh^2 \xi} \overline{n} \tag{3.19}$$

and the corresponding uncertainty relation gives

$$(\Delta U_1)^2 \cdot (\Delta U_2)^2 = \left(\frac{\overline{n}}{4 \sinh^2 \xi} \right)^2 \tag{3.20}$$

with $\overline{n}$ equal to the equation (3.8).

From equation (3.18) to (3.20) we see that when $q \to 1$,

$$(\Delta U_1)^2 = \frac{1}{4} \frac{(1 + \tanh \xi)^2}{\tanh^2 \xi} \sinh^2 \xi = \frac{1}{4} e^{2\xi} , \tag{3.21}$$

$$(\Delta U_2)^2 = \frac{1}{4} \frac{(1 - \tanh \xi)^2}{\tanh^2 \xi} \sinh^2 \xi = \frac{1}{4} e^{-2\xi}, \tag{3.22}$$

and

$$(\Delta U_1)^2 \cdot (\Delta U_2)^2 = \frac{1}{16}. \tag{3.23}$$

So the fluctuations of two quadrature reduced to minimum uncertainty relation as expected.

4. Thermal entanglement of photons

One can also discuss the properties of photons in the q-deformed thermal states. To achieve this, a q-deformation of the thermal state have to be developed. The crucial point is to define q-deformed vacuum $|O(\xi)\rangle$. The q-deformed thermal vacuum $|O(\xi)\rangle$ can be defined as [9]

$$|O(\beta)\rangle = \sum_{n=0}^{\infty} P_n^{\frac{1}{2}} |n\widetilde{n}\rangle , \tag{4.1}$$

where the doubled q-Fock state $|n\widetilde{n}\rangle$ is given by the equation (2.22) and

$$P_n = \frac{e^{-\beta E_n}}{Z(\beta)} \tag{4.2}$$

denotes probability of the state with energy E_n and $Z(\beta)$ is the partition function. The thermal vacuum is normalized as

$$\langle O(\beta)| \, |O(\beta)\rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_m^{\frac{1}{2}} P_n^{\frac{1}{2}} \langle m\widetilde{m}| \, |n\widetilde{n}\rangle = \sum_{n=0}^{\infty} P_n = 1. \tag{4.3}$$

Once that the q-thermal vacuum is defined, we can define further q-thermal creation and annihilation operators through Bogoliubov transformation. By using the creation and

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annihilation operators on the q-thermal vacuum, a complete orthonormal set of state vectors containing $|O(\xi)\rangle$ as one of the members can finally be found. This process is completely parallel to the corresponding process in undeformed thermal field dynamics [11].

Consider now the average photon number in the q-thermal vacuum. Analogue to the usual case, the average photon number can be found as

$$\overline{n} = \langle O(\beta)| a^+ a |O(\beta)\rangle. \tag{4.4}$$

Note that the operators a, a^+ in the equation (4.4) satisfy the q-deformed commutation relation, the equation (2.20) in the Biedemharn and Mackfalan's deformation scheme.

For the photons (bosons) system, the thermal vacuum $|O(\beta)\rangle$ is given by equation (4.1) with probability

$$P_n = \frac{\exp(-\beta E_n)}{Tr(e^{-\beta H})} = [1 - \exp(-\beta\omega)]\exp(-n\beta\omega). \tag{4.5}$$

In the equation (4.5), $\beta = \frac{1}{kT}$ with Boltzmann constant k and temperature T. The free Hamiltonian H of the system is given by [7]

$$H = \omega N. \tag{4.6}$$

Substituting equation (4.5) into the equation (4.4) and by using equation (2.18) and $D_q(n)$ in the Biedonharn and Mackfalan's scheme we get

$$\overline{n} = \left\{ \frac{C_1}{\exp\{\beta(\omega + \frac{\lambda}{\beta})\} - 1} + \frac{C_2}{\exp\{\beta(\omega - \frac{\lambda}{\beta})\} - 1} \right\} \tag{4.7}$$

with $\lambda = \ln q$ and

$$C_1 = \frac{q - 1}{q - q^{-1}}, \quad C_2 = \frac{1 - q^{-1}}{q - q^{-1}}. \tag{4.8}$$

$$C_1 + C_2 = 1. \tag{4.9}$$

We see from the equation (4.7) that the distribution function (average photon number) of the photons now divided into two part: one with energy $\omega + \frac{\lambda}{\beta}$ and the other with energy $\omega - \frac{\lambda}{\beta}$ due to the q-deformation of commutation relation. In the equation (4.7), in order to guarantees the weights $C_1, C_2 > 0$ and $C_1 + C_2 = 1$, $q > 0$ has to be assumed.

It is easy to check that when $q \to 1$, the photon number distribution turn out to be

$$\overline{n} = \frac{1}{\exp(\beta \, \omega) - 1} \qquad (4.10)$$

as we expected.

Analogue to the section 3, one can also discuss the fluctuations of two quadrature in thermal state. The fluctuations are given by

$$(\Delta U_i)^2 = \langle O\left(\beta\right)| \, U_i^2 \, |O\left(\beta\right)\rangle - \left(\langle O\left(\beta\right)| \, U_i \, |O\left(\beta\right)\rangle\right)^2 . \qquad (4.11)$$

Straightforward calculations gives the expectation values

$$\langle O\left(\beta\right)| \, a^+ a \, |O\left(\beta\right)\rangle = \overline{n}, \qquad (4.12)$$

$$\langle O\left(\beta\right)| \, a a^+ \, |O\left(\beta\right)\rangle = e^{-\beta\omega} \cdot \overline{n} \, , \qquad (4.13)$$

$$\langle O\left(\beta\right)| \, a\tilde{a} \, |O\left(\beta\right)\rangle = e^{-\frac{1}{2}\beta\omega} \cdot \overline{n} \, , \qquad (4.14)$$

$$\langle O\left(\beta\right)| \, a^+ \tilde{a}^+ \, |O\left(\beta\right)\rangle = e^{-\frac{1}{2}\beta\omega} \cdot \overline{n} \qquad (4.15)$$

and the others are zero. Substituting these results to the equation (4.11) we obtain

$$(\Delta U_1)^2 = \frac{1}{4} \left(e^{\frac{1}{2}\beta\omega} + 1 \right)^2 \overline{n} \, , \qquad (4.16)$$

$$(\Delta U_2)^2 = \frac{1}{4} \left(e^{\frac{1}{2}\beta\omega} - 1 \right)^2 \overline{n} \, , \qquad (4.17)$$

and the corresponding uncertainty relation gives

$$(\Delta U_1)^2 \cdot (\Delta U_2)^2 = \frac{1}{16} \left(e^{\beta\omega} - 1 \right)^2 \overline{n}^2. \qquad (4.18)$$

when $q \to 1$, the average number of photon $\overline{n} \to \frac{1}{\exp(\beta \, \omega) - 1}$ and

$$(\Delta U_1)^2 \big|_{q \to 1} = \frac{1}{4} \frac{e^{\frac{1}{2}\beta\omega} + 1}{e^{\frac{1}{2}\beta\omega} - 1}, \qquad (4.19)$$

$$(\Delta U_2)^2 \big|_{q \to 1} = \frac{1}{4} \frac{e^{\frac{1}{2}\beta\omega} - 1}{e^{\frac{1}{2}\beta\omega} + 1}. \qquad (4.20)$$

Thus we find

$$(\Delta U_1)^2 \cdot (\Delta U_2)^2 \big|_{q\to1} = \frac{1}{16}, \tag{4.21}$$

namely, the fluctuations satisfy minimum uncertainty relation.

From above discussions we see that the behavior of photons in the q-thermal state is completely same with the behavior of photons in the q-squeezed states. The only difference is the factor $tanh\xi$ in the q-squeezed system is displaced by the factor $e^{-\frac{1}{2}\beta\omega}$ in the thermal system. The thermal vacuum defined in equation (4.1) is an infinite superposition of doubled number state $|n\widetilde{n}\rangle$ constructed by direct product of single-mode states $|n\rangle$ and $|\widetilde{n}\rangle$. So we conclude that the thermal vacuum is also a kind of entangled state, the thermally entangled state. The entanglement of the system can be found as

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\begin{aligned}
E &= -\sum_{n=0}^{\infty} P_n \log_2 P_n \\
&= -[1 - \exp(-\beta\omega)]\exp(-n\beta\omega)\log_2\left\{[1-\exp(-\beta\omega)]\exp(-n\beta\omega)\right\} \\
&= -\left\{\overline{n_0}\log_2\overline{n_0} - (1+\overline{n_0})\log_2(1+\overline{n_0})\right\},
\end{aligned} \tag{4.22}
$$

where $\overline{n_0} = \frac{1}{\exp(\beta\,\omega)-1}$ denotes the average number of photons in the undeformed thermal state. We see that the entanglement of photons is depended on the parameter $\beta = \frac{1}{kT}$, the temperature of the system.

5. Conclusions and Discussions

We constructed a doubled q-Fock space by introducing an idle system dual to the physical one under consideration and investigated the effects of q-deformation of Heisenberg-Weyl algebras to the average numbers and quantum fluctuations in the squeezed state and thermal state. The results show that the q-deformation decompound the photon distribution into two parts and leads to energy shift $\pm\frac{\gamma}{\beta}$ in the thermal state. We pointed out that the new vacuums, the squeezed vacuum $|O\,(\xi)\rangle$ and thermal vacuum $|O\,(\beta)\rangle$ in the doubled q-Fock space are entangled states of two-mode with infinite degree of freedoms.

It is worthy to discuss that what is the meaning of the idle (tilde) system and why we must construct a doubled space in order to define the new vacuums such as squeezed vacuum $|O\,(\xi)\rangle$ or thermal vacuum $|O\,(\beta)\rangle$. We will discuss these questions in the forthcoming works [12].

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